REFLECTION ARTICLE

Characterization of the Mathematics Teacher's Specialized Knowledge

Caracterización del conocimiento especializado del profesor de matemáticas

Caracterização do conhecimento especializado do professor de matemática

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OPEN ACCESS

DOI: http://dx.doi.org/10.18634/sophiaj.18v.2i.1175

Article Information

Received: March 2022
Revised: June 2022
Accepted: September 2022
Published: December 2022

Keywords: professional knowledge, specialized knowledge, mathematics teacher, MTSK model.

Palabras clave: conocimiento profesional, conocimiento especializado, profesor de matemáticas, modelo MTSK.

Palavras-chave: conhecimento profissional, conhecimento especializado, professor de matemática, modelo MTSK.

How to cite:
Padilla Escorcia, Iván, & Acevedo-Rincón, J. P. (2022). Characterization of the Mathematics Teacher's Specialized Knowledge. Sophia, 18(2). https://doi.org/10.18634/sophiaj.18v.2i.1175

Sophia-Education, volume 18 number 2. July/December. English version.

ABSTRACT

The main feature of the MTSK model is to study the mathematical and didactic-pedagogical knowledge required by a mathematics teacher to teach this area of knowledge at any academic level, which has never before been explicitly described in the models related to the study of the mathematics teacher, with respect to the knowledge required by a teacher for the disciplinary teaching of mathematics, which differentiates him/her from other types of professionals with knowledge of the area. In this order, the structure of the MTSK model is presented, expressed in domains, subdomains, categories of subdomains and examples of the categories of the subdomains that contribute to the exploration of the knowledge that the mathematics teacher must have to teach mathematics contents. It is concluded that, given the versatility of this model, it can be taken as a reference for the constitution of new lines of research in the field of mathematics teacher training.

RESUMEN

En este artículo de reflexión tiene como objetivo caracterizar el conocimiento especializado del profesor de matemáticas, desde una mirada del modelo MTSK, cuyo rasgo principal consiste en estudiar el conocimiento matemático y didáctico-pedagógico que requiere un profesor de matemáticas para enseñar esta área del conocimiento en cualquier nivel académico y que nunca antes en los modelos relacionados con el estudio del profesor de matemáticas había sido descrito de una forma explícita, con respecto al conocimiento que requiere un profesor para la enseñanza disciplinar de las matemáticas, que lo hace diferenciar de otro tipo de profesionales, con conocimiento del área. En ese

ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

Conflicto de interés:

Los autores declaran no tener ningún conflicto de interés.

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orden, se presenta la estructura del modelo MTSK expresada en dominios, subdominios, categorías de subdominios y ejemplos de las categorías de los subdominios que contribuyen en la exploración del conocimiento con el que debe contar el profesor de matemáticas para enseñar contenidos de las matemáticas. Se concluye que, dada la versatilidad de este modelo, puede ser tomado de referencia para la constitución de nuevas líneas de investigación en el campo de la formación del profesor de matemáticas.

RESUMO

Neste artigo de reflexão, o objetivo é caracterizar o conhecimento especializado do professor de matemática, sob a ótica do modelo MTSK, cuja principal característica consiste em estudar os conhecimentos matemáticos e didático-pedagógicos que um professor de matemática requer para ensinar esta área da conhecimentos em qualquer nível acadêmico e que nunca antes nos modelos relacionados ao estudo do professor de matemática foram descritos de forma explícita, no que diz respeito aos conhecimentos que um professor requer para o ensino disciplinar da matemática, o que o torna diferente de outros tipos de profissionais, com conhecimento da área. Nessa ordem, é apresentada a estrutura do modelo MTSK expressa em domínios, subdomínios, categorias de subdomínio e exemplos das categorias de subdomínio que contribuem para a exploração do conhecimento que o professor de matemática deve ter para ensinar o conteúdo de matemática. Conclui-se que, dada a versatilidade desse modelo, ele pode ser tomado como referência para a constituição de novas linhas de pesquisa no campo da formação de professores de matemática.

Introduction

Over the years, many changes have taken place in education; different models, theories, approaches and strategies have been applied in the teaching-learning processes, generating impact, repercussion and contribution in teachers' practices. In this order, Schön (1983) is considered one of the precursors in studying the professional knowledge of a teacher who teaches a disciplinary knowledge in a general way. According to this author, the knowledge of a teacher is evidenced from the reflection he/she makes from the experience of his/her pedagogical practice. In the context of mathematics education, Climent (2005) states that the professional knowledge of a mathematics teacher is situated and contextualized, i.e., the teacher develops his/her knowledge according to the environment where he/she is and not individually, but as a group and shared with the community. However, the professional knowledge of one teacher is different from that of another, since it is based on the individual beliefs and conceptions with which the teacher lives (Schoenfeld, 2010).

Thus, characterizing the specialized knowledge of a licensed professional is one of the most complex challenges that education has gone through. In this sense, it seems necessary that a teacher should not only have knowledge in the area of knowledge of which he/she is a professional, but also have pedagogical-didactic knowledge that allows him/her to create harmony in his/her pedagogical practices. Therefore, Shulman (1986) divided the teacher's knowledge into three categories: knowledge of the content of the subject, pedagogical knowledge of the content, and curricular knowledge, which are listed below.

Knowledge of subject matter content is that which goes beyond the concepts of a subject matter, that is, knowledge of the structures of the subject matter itself. These structures include substantive and syntactic structures; substantive structures are the teacher's knowledge to incorporate the principles and concepts of the discipline that are applied to different facts of daily life. For example, the teacher's knowledge to apply the concept of multiplication of natural numbers that facilitates the conversion of dollars to Colombian pesos; on the other hand, syntactic structures are the teacher's knowledge about the formal and demonstrative part of the contents of the discipline, determined by their level of falsity, truth, validity, or invalidity. For example, the teacher's knowledge to demonstrate why in the properties of potentiation, every number raised to zero is equal to 1, given the transitivity relation.

$$\frac{a^m}{a^m} = a^{m-m} = a^0 \text{ y } \frac{a^m}{a^m} = 1, \text{ por lo tanto, } a^0 = 1$$

On the other hand, pedagogical knowledge of the content is the knowledge that is linked to knowledge of the content from the teaching dimension. This type of knowledge is composed of elements such as: forms of representation, analogies, illustrations, examples, explanations, strategies and demonstrations used by the teacher to make the content more understandable to the students (Shulman, 1986). For example, in the case of teaching strategies, the teacher's knowledge of how to represent functions in specialized mathematical software such as GeoGebra, allows students to have access to the analysis of the properties of functions in the plane such as: domain, range, period, points of intersection, among others. This is because one of the main functionalities of these software is to analyze in a deep and robust way what happens in the Cartesian plane (Padilla-Escorcia and Acevedo-Rincón, 2021).

Curriculum knowledge is defined as the teacher's knowledge of the programs designed for the teaching of specific subjects and contents of a given area of knowledge at a certain academic level (Shulman, 1986). In addition, it includes knowledge of the curriculum itself and the various materials proposed in it for teaching, as well as the treatments that may exist to guide a subject, sequence it or the forms of evaluation proposed in the same curriculum (Montes, 2015).

Thus, although Shulman (1986) did not address his model specifically from the context of mathematics, it was of great help for mathematics educators worldwide to decide to study on the professional knowledge of mathematics teachers. This, seen from the viewpoint of Montes et al. (2020), enriches the researcher's interpretation of the nature of mathematics teachers, thus making it possible to evaluate their qualities and to study how this professional knowledge is related through the study of the teacher's competencies.

Some of the most prominent models are presented in this review article, such as: Bromme (1994), Fennema and Franke (1992) and Ball et al. (2008), this because they are models that study the mathematics teacher and that are cited in the MTSK model (Carrillo et al., 2013; Carrillo et al., 2018), model in question to be studied in this paper.

Bromme's Model

Bromme (1994) was one of the first to address the professional knowledge of the mathematics teacher through five domains that make up his model and that are closely related to that proposed by Shulman (1986): knowledge of the content of mathematics as a discipline, knowledge of school mathematics, philosophy of school mathematics, pedagogical knowledge, and subject-specific pedagogical knowledge, which are defined below.

Content knowledge of mathematics is the learning that teachers obtain during their academic studies, such as, for example, propositions, rules, mathematical ways of thinking and methods, which are visible in mathematician training manuals.

The knowledge of school mathematics is the teacher's ability to understand that the contents to be taught are not simply the pure basics of the subject, but rather that they are adapted to the reality experienced in schools.

The philosophy of school mathematics is the teacher's knowledge of the epistemological foundations of mathematics, as well as the learning of mathematics, or the relationships between mathematics, human life and other disciplines.

In terms of pedagogy, Bromme (1994) established two domains, one general and the other specific. The general one is the knowledge of guidelines on how to maintain a group work environment. For example, the strategies that a teacher uses to maintain discipline and order in class (such as the relaxation game and the applause game). On the other hand, specific pedagogical knowledge is that which is integrated or mixed between pedagogical knowledge and the teacher's professional experience with mathematics, because

ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

it is not the same thing to have a specific pedagogical knowledge. because it is not the same for a mathematics teacher whose professional experience is null (recently graduated) with that of a mathematics teacher with years of experience, since it is expected that he/she has gone through experiences that have strengthened his/her practice as a teacher, which in turn allows a deeper relationship with his/her pedagogical knowledge.

The Fennema and Franke Model

Fennema and Franke (1992) focused on the fact that the teaching of mathematics content should be dynamic and interactive. Therefore, they considered that in the "teacher knowledge" model, it is necessary for teachers to have sufficient knowledge about mathematics as a discipline to be able to teach it. To this end, they established four domains in the model: content knowledge, didactic knowledge, knowledge of students' cognition and teachers' beliefs. These four domains are defined as follows:

(i) content knowledge is that in which the concepts, procedures and processes in solving problems in environmental situations that relate mathematics are understood, as well as the interrelationships that occur between concepts and procedures in this type of situations; (ii) didactic knowledge consists of the type of procedures, organization and motivational techniques that teachers use as strategies in the planning of class units; (iii) knowledge of students' cognition is the teachers' knowledge to understand students' learning processes, that is, what is easy and difficult for them in the different mathematics contents; (iv) teachers' beliefs are the perceptions that influence their understanding of mathematics teaching, depending on the context in which they find themselves and the experiences they have; (v) teachers' beliefs are the perceptions that influence their understanding of mathematics teaching, depending on the context in which they find themselves and the experiences they have.

The Ball and Collaborators' Model

In that same decade, Deborah Ball and her collaborators, in the educational mathematics research group at the University of Michigan in 2008, refined the model proposed by Shulman in the 1980s. Their concern was to study the knowledge of the mathematics teacher, with the idea of refining this profession, so that marked differences could be established between mathematics teachers and other professionals who also have knowledge in this area. For this, the domains proposed by Shulman (content knowledge and didactic content knowledge) were the basis for this model called "Mathematical Knowledge for Teaching" (Ball et al., 2008), each with three respective subdomains.

Regarding content knowledge, Ball et al. (2008) proposed the following three subdomains: common content knowledge, specialized content knowledge, and content horizon knowledge, which are defined as follows: (i) Common content knowledge is the knowledge used in situations that are not exclusive to teaching, i.e., that any person trained in mathematics could possess. For example, the notion of knowing when a fraction with equal denominator is greater or less than another through quantity and partition analysis; (ii) Specialized knowledge, which is the knowledge of the purely mathematical skills required by a teacher for teaching and which is considered the most significant contribution highlighted in this model, since it led to think about the need for mathematics teaching to be in charge of professionals with training in Mathematics Education and that some of the most recurrent tasks that a teacher with specialized knowledge in mathematics should fulfill were: presenting mathematical ideas, finding examples to address a specific mathematical topic, relating a topic to another from previous or later years, modifying tasks to make them more complex or simple, evaluating students' arguments, using the formal language of mathematics in definitions, among others (Montes, 2015); (iii) Knowledge of the mathematical horizon is the teachers' level of awareness of how mathematics contents are distributed in the curriculum, so that he/she is able to establish relationships between contents depending on the different levels of schooling; these can also be within the same concept (intra-conceptual) or between different concepts (inter-conceptual). For example, an elementary school teacher must know the knowledge and interpretations required by his students about fractional numbers in order to deal with rational numbers in the secondary grades (interconceptual connection).

Regarding didactic content knowledge, Ball et al. (2008) proposed the following three subdomains: knowledge of content and students, knowledge of teaching content, and curricular knowledge, which are stated as follows: (i) Knowledge of content and students is the knowledge that blends knowing students and mathematics content from difficulties, obstacles and needs faced by students in learning the topics. For example, the knowledge that the teacher has to identify that the main causes that lead to poor performance of students in learning Algebra are derived from difficulties in operating natural, integer and rational numbers; (ii) The knowledge of teaching content is defined as the mixture of knowing the teaching and the contents of mathematics, this to design strategies, didactic units and activities that allow facilitating the learning of students based on the errors they have. For example, in the teaching of fractions, using real-life elements such as cookies, apples, chocolate, among others, and dividing them into equal parts are situations that allow understanding the essence of the fraction concept; (iii) Curricular knowledge is the knowledge about the curricular objectives and goals that guide the pedagogical practice of the mathematics teacher, being very similar to what was proposed in the 1980s by Shulman (1986).

However, it should be noted that one of the main difficulties encountered in this model was in determining what is considered as common knowledge and specialized knowledge in mathematics, since according to the level of schooling at which a teacher is teaching mathematics, it could be stated whether he or she has specialized knowledge of a specific content (Flores, Sosa, & Ribeiro, 2016). For example, an ordinary person by intuition might know that the decimal number 2.405 is less than the decimal number 2.41, however, a teacher with specialized knowledge of the subject knows, that 2.405 is less than 2.41, because in the comparison of the hundredths of the decimal number 0 is a number less than 1.

The model proposed by Ball et al. (2008), although it offered greater clarity and explicitness in detailing the knowledge required by the teacher for teaching mathematics, addressing aspects such as knowledge of teaching and the teacher's knowledge of his students' learning, in terms of the didactic-pedagogical domain. It is considered that the subdomain of specialized knowledge of the teacher, which was proposed as a way to differentiate it from the common knowledge of a professional with mathematical knowledge, was left open to the interpretation of the reader and the literature. It is clear that Ball and collaborators thought of this subdomain as something more of a mathematical type, so it would have been interesting to look at it from the didactic-pedagogical knowledge, which is expected to be achieved by mathematics graduates, and not professionals such as mathematicians, physicists, engineers, among others, whose training does not contemplate the aforementioned. Likewise, in this subdomain, beyond the fact that it has been thought purely at the disciplinary level, it did not explicitly show in what knowledge of mathematics someone with specialized knowledge differs from someone with common knowledge.

Mathematics Teacher’s Specialized Knowledge (MTSK)

The model called Mathematics Teacher Specialized Knowledge (MTSK), in Spanish "Conocimiento especializado del profesor de matemáticas" is generated within the group "Seminario de Investigación en Educación Matemática" (SIDM) and coordinated from the University of Huelva in Spain. According to Carrillo et al. (2018). This model, arises from a need to complement work done by Fennema and Franke (1992), Bromme (1994), Ball et al., (2008), among others, regarding the professionalization of the mathematics teacher, as well as an opportunity to be a trend within new lines of research that were aimed at the analysis of the mathematics teacher, as an opportunity for improvement for their pedagogical practice. That is why this model focuses on the professional knowledge that the teacher needs and uses to explain and understand the nature of mathematics. For this reason, it does not cover the general pedagogical knowledge of teachers (pedagogy and psychology) since this knowledge is not specific to mathematics.

This model consists of two domains. On the one hand, the **Knowledge of Mathematics** (MK), which represents a network of knowledge structured according to rules and connections that allow understanding the nature of mathematics, the reason and origin of the procedures, the mathematical language and its precision; In addition, it encompasses the mathematical knowledge that the teacher uses, or can use, in any activity and that transcends even more than the mathematical content that a student of the academic level in which he/she teaches is intended to learn, not only in quantity of knowledge but also in the nature of this, that is, in the diverse applications that the contents of mathematics have (Carrillo et al., 2018; Advíncula et al., 2021). From this, the following three subdomains emerge: knowledge of topics (KoT), Knowledge of the Structure of Mathematics (KSM) and knowledge of the Practice of Mathematics (KPM).

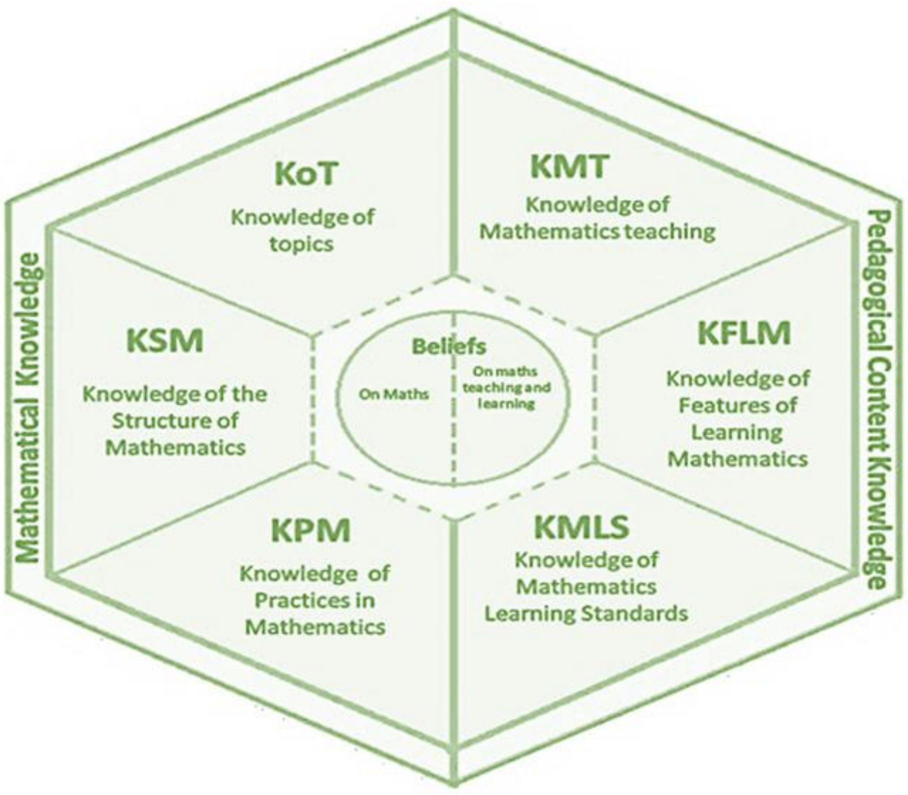
On the other hand, the **pedagogical content knowledge** (PCK) domain, which is based on the ways of deepening the mathematical content when there is the intention of teaching and learning (Rojas, Flores-Medrano, & Carrillo, 2015; Vasco & Climent, 2018). In turn, the following three subdomains emerge: Knowledge of

ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

Features of Learning Mathematics (KFLM), Knowledge of Mathematics Teaching (KMT), and Knowledge of Mathematics Learning Standards (KMLS).

In turn, each of the six subdomains mentioned above, revolve around the beliefs and conceptions that teachers have based on the experiences they have in teaching mathematics, and which are identified as the philosophy of mathematics, as a component within the teacher's body of knowledge (Carrillo et al., 2017). Thus, Figure 1 shows how the knowledge domains and subdomains of the model are distributed. On the left side of the model is located the Mathematical knowledge (MK) and its 3 subdomains: Knowledge of topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of the Practice of Mathematics (KPM); on the other hand, on the right side of the model, is located the didactic-pedagogical content knowledge domain (PCK) and its 3 subdomains: Knowledge of Mathematics teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS). In the center of the model, the teacher's beliefs and perceptions of and how these underlie each knowledge subdomain are located, as seen in Figure 1:

Figure 1. Mathematics teacher's specialized knowledge model



Source: Carrillo et al. (2018).

However, the KoT, **knowledge of the topics**, does not only refer to the knowledge of mathematics as a discipline, but also includes school mathematics. That is, it describes what and how the mathematics teacher knows about the topics to be taught, the knowledge of mathematical contents (concepts, procedures, facts, rules, theorems, lemmas, etc.) and their meanings, as well as the connections established between the different contents and the ways of using the representation registers (Escudero and Carrillo, 2020). In the KoT, five categories are envisioned according to what was contributed by Flores-Medrano et al. (2014) in their doctoral thesis and which are indispensable for every teacher who teaches mathematics.

1. The *phenomenology* of content corresponds to the teacher's knowledge about the use and applications of a specific topic within mathematics itself (intra-conceptual), as well as about real-life situations within which the teacher can place a topic (Vasco, Climent, Escudero, Montes & Ribeiro, 2016). For example, the teacher's knowledge to identify that the representation of the transmission of a disease or the draining of a tank can be studied by using a differential equation.
2. The category *properties and fundamentals attributable to a particular topic* corresponds to the teacher's knowledge about the use of properties and rules of particular content in mathematics. For example, the teacher's knowledge about the properties of potentiation that must be taken into

account to solve an algebraic expression with powers. This is observed in the following operation combined with powers $X^3X^2X^6/X^7$; in this one, using the properties of powers independently contributes to solve the exercise in a faster and simpler way, initially applying the product property of powers of equal bases the result would be X^{11}/X^7 , then the quotient property of powers of equal bases that gives as a result X^4 . Likewise, the difference between properties depending on the mathematical structure in which a content is being addressed; for example, matrices do not comply with some properties of algebraic structure, such as the commutative property, which do comply with the basic operations of addition and multiplication in real numbers.

3. *Representational* registers are the teacher's knowledge of the different ways in which a content can be represented, either graphically, verbally, numerically, analytically, etc. For example, sets that can be represented by comprehension or by extension. Thus: A= {the set of odd numbers less than 21} or A= {1, 3, 5, 7, 7, 9, 11, 11, 13, 15, 17,19}; likewise occurs in fractions, which can be represented graphically, concretely, verbally or symbolically.
4. *Definitions* are the teacher's knowledge to, according to the properties that are met in mathematics, define a specific topic. For example, the definition of an odd number, considering that it must be of the form 2+1 or 2+3, the definition of a right triangle, since it must have at most one right angle (90°).
5. The category of *procedures* is the professor's knowledge about the conventional and alternative algorithms that are used in the contents of the mathematical; it is the knowledge that has the professor to answer questions that formulate students as: How do you do? Why do you do so? Why do you use so? or What do you mean that? An example of this is the teacher's knowledge of why the range of the trigonometric function established by the formula is $f(x) = -\frac{1}{2} sen\, x$ es $(\frac{1}{2}, -\frac{1}{2})$ and the range of the function $g(x) = 4\, sen\, x$ is (-4, 4), among other examples proposed by Aguilar-González, Muñoz-Catalán and Carrillo (2019) about the teacher's knowledge of the properties and elements (angles, sides, and vertices) that are used to define polygons in Geometry.

The KSM, **Knowledge of Structures of Mathematics**, corresponds to the personal construct that the teacher develops about the way in which mathematics topics are internally connected in order to relate them to each other, either from the course he/she is teaching or with contents from other courses or higher levels (Montes, 2015; Vasco and Climent, 2018), so that these relationships allow increasing or simplifying the degree of complexity of a specific topic (inter-conceptual and intra-conceptual connections), that is, within the same field of mathematics (Flores-Medrano et al., 2016). Now, inter-conceptual connections are mathematical ideas that link representations of the same or different concepts; intra-conceptual connections are mathematical ideas that take place in the proximity of a single concept (Martínez et al., 2011). Likewise, Flores-Medrano et al. (2014) propose four categories about the mathematics teacher's knowledge to establish connections in the contents he or she teaches; these are: complexification, simplification, transversal and auxiliary connections.

1. *Complexity connections* are the teacher's knowledge to relate the contents taught with subsequent contents, that is, a vision of elementary mathematics from an advanced point of view from the particular to the general. For example: the teacher's knowledge to relate that the teaching of factorization will be fundamental in the learning of the limit of functions; likewise, the teacher's knowledge to relate the geometric notions that students develop (line, point, angle, polygons) in elementary school, with the demonstration of Euclidean geometry theorems in secondary school.
2. *Simplification connections* are the teacher's knowledge to relate the contents he/she teaches with previous contents, i.e. previously addressed by his/her students. For example, in the realization of the derivative of the function expressed by the formula $f(x) = \frac{x^2-1}{x^2+2x+1}$, the teacher's knowledge of factoring, both the numerator and the denominator, is denoted; in the case of the numerator: (x^2-1) como $(x-1)(x+1)$, y en el denominador (x^2+2x+1), como $(x+1)$.

ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

which is considered as an interconceptual connection. In this case the function reduces to $f(x) = \frac{x-1}{x+1}$, which is much simpler to derive than as originally proposed.

3. The *connections of transversal contents* are the teacher's knowledge to relate common contents of different types of mathematical thinking, such as, for example, the limit, the derivative, and the point and global continuity that underlie infinite processes, or also the relation mentioned by Climent et al. (2021) that equality is present in numerical and algebraic expressions (numerical thinking) and that it is related in turn to the congruence of geometric figures and similarity (geometric thinking).

4. Finally, *auxiliary connections* are the teacher's knowledge to make inter-conceptual connections that are not part of the content being taught and that contribute to the solution of the problem being developed. For example, in the limit, since this is an operation, whose result is equal to an indeterminacy of the form: if the value of x is replaced, L'Hopital's rule is used as an auxiliary connection, since, by means of the concept of derivative, both the numerator and denominator of the function are derived, until the indeterminacy disappears in this one.

On the other hand, KPM, **Knowledge of Practices in Mathematics**, is the specific knowledge of the teacher to demonstrate, justify, validate, make deductions and inductions, and generate knowledge in mathematics, that is, its pure meaning. Likewise, it is the teacher's knowledge about the hierarchical organization, the ways to proceed in solving mathematical problems, the good use of formal symbols in mathematics, and the necessary and sufficient conditions required to generate a definition in mathematics. In addition, the teacher's knowledge about particular practices in mathematics such as modeling (Montes, 2015; Aguilar-González, Muñoz-Catalán, & Carrillo, 2019; Flores-Medrano et al., 2016; Padilla-Escorcia & Acevedo-Rincón, 2020; and Zakaryan & Sosa, 2021).

Now, Flores-Medrano et al. (2014) propose categories that are linked to the knowledge of mathematics teachers' practices. These are: practices linked to mathematics in general and practices linked to a subject in mathematics, which are defined as follows:

1. *Practices linked to mathematics in general* are the teacher's knowledge of how mathematics develops independently of the concept being worked on and of the logical structures of thought that contribute to understanding the functioning of multiple mathematical aspects; for example, the teacher's knowledge that points of accumulation, points of adhesion, or interior points are concepts that, from their abstraction, are derived from concepts of general topology.

2. *Practices linked to a subject in mathematics* are the teacher's knowledge about the contents to be taught from its mathematical structure; for example, knowing the meaning of a necessary condition and a sufficient condition to work generically in mathematics, which allows the creation of logical structures of thought that can improve the understanding of everyday phenomena (Montes, 2015).

The KFLM, **knowledge of Features of learning Mathematics**, is focused on recognizing the contents of mathematics as learning objects (Escudero and Carrillo, 2020; Aguilar-González et al., 2018), that is, the teacher's knowledge to understand the relationships that students establish with the concepts worked on, as well as the capabilities and weaknesses they have about them. That is, that the teacher knows how the student learns and how the students develop cognition according to the topics addressed. Likewise, the knowledge of the obstacles, difficulties, and errors that students present the most when learning the contents of mathematics.

However, Escudero-Ávila and Carrillo (2020) propose that this subdomain of the MTSK can be broken down into the following categories: strengths and difficulties associated with learning mathematical content, forms of student interaction with mathematical content, student interests and expectations, and learning theories associated with mathematical content.

1. *Knowledge of the strengths and difficulties associated with the learning of mathematical content* is the teacher's knowledge of the misconceptions that exist about a given subject on the part of a student and that are specific to the specific mathematical content, i.e., that they are directly associated with the

mathematical and non-pedagogical characteristics of the content. Added to this, in this category is also the knowledge of incorrect mathematical ideas that students may acquire from a mathematics content. An example of this is the difficulties that students have in interpreting graphically the concept of range of the trigonometric sine function, because they do not understand the concept of range of a function in general.

2. *Knowledge of students' ways of interacting with a mathematical content* refers to the teacher's knowledge of students' processes and strategies, both typical and unusual in learning mathematics; it also includes the teacher's knowledge of students' possible modes of knowledge construction associated with the very nature of the mathematical content. An example of this is the teacher's knowledge that the strategy students use to add or subtract heterogeneous rational numbers is the "smiley face" or cross method over the least common multiple method, even when it is more than three rational numbers that are being operated on.
3. Regarding *knowledge of students' interests and expectations about mathematical content*, it is the teacher's knowledge about students' expectations and interests with respect to learning mathematics, in addition to knowledge about the preconceptions of ease or difficulty commonly associated with the different areas of mathematics that students count on.
4. *Learning theories associated with a mathematical content* is the teacher's knowledge of the possible forms of apprehension associated with the nature of the mathematical content. It includes the knowledge of theories about the student's cognitive development, both for mathematics in general and for particular content, which come from the teacher's professional experience or from research supported by theories that allow explaining the processes of construction of mathematical knowledge from the perspective of teaching and learning (Escudero-Ávila, 2015).

On the other hand, as for KMT, **knowledge of mathematics teaching**, it is the teacher's knowledge about formal theories of mathematics teaching, derived from research in Mathematics Education or from observations and reflections of mathematics activities in the classroom. This focuses on the skills and degree of awareness that the teacher possesses in the selection and use of strategies at the conceptual or pedagogical level for teaching mathematics (Delgado and Zakaryan, 2019; Montes, 2015). In that order, Flores et al. (2014) proposed three categories that characterize the teacher's knowledge from the viewpoint of mathematics teaching, these are: personal and institutionalized theories of teaching, knowledge about material and virtual resources, and knowledge of strategies, techniques, examples, and tasks for teaching mathematical content, which are defined as follows:

1. *Personal and institutionalized teaching theories* are the teacher's knowledge about teaching theories in Mathematics Education, such as, for example, the theory of didactic situations proposed by Brousseau and the situations that can occur in the same [action, formulation, validation, and institutionalization] and that serve to plan activities or strategies in class. In addition, the knowledge to identify the level of potentiality offered by this type of activities, workshops and didactic units proposed, or the knowledge of exercises, metaphors, explanations, and analogies that the teacher knows and influence the learning and interpretation of students in the contents. For example, the teaching of prime numbers using Eratosthenes' sieve in concrete material (cardboard).
2. *Knowledge about material and virtual resources* is the knowledge to teach mathematics through textbooks, rulers, normal and electronic blackboard, tangram, specialized software of the area, among others.
3. The *knowledge of strategies, techniques, examples and tasks for teaching mathematical content* is the knowledge that the teacher has about the potential offered by activities, tasks, examples, strategies (includes the use of analogies for teaching) or didactic techniques in the teaching of mathematics content (Espinosa-Vásquez, Zakaryan, & Carrillo, 2018). An example of this would be when a teacher teaches frequency distribution tables in statistics and proposes a task in which it is the students themselves who collect the information to be tabulated by means of a survey and a topic that is of interest to them. On the other hand, in the category of examples, this can become visible when a teacher proposes different exercises to explain the range of the trigonometric sine function, for example, $f(x) = sen(x+3), f(x) = sen(x+7), f(x) = sen(x-7)$ o $f(x) = sen(x+9)$,

ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

or whose behavior depends on the constant accompanying the argument of the function with form $\lim_{x\to 0}\frac{1-cosx}{x^2}$ in this case *a*.

The KMLS, **knowledge of mathematics learning standards**, corresponds to the teacher's knowledge about the learning outcomes he/she expects his/her students to have depending on the school level in which they are, in addition to the level of deepening with which a mathematics content should be focused based on school levels and national and international learning standards and indicators, which indicate the organization of the topics based on previous courses (Muñoz-Catalán, Liñan, & Ribeiro, 2017). In this sense, Flores-Medrano et al. (2014) presented a series of categories that deepen the specialized knowledge of the teacher in this subdomain. These are: Knowledge that the teacher has about the mathematical content required in the school grade in which classes are being taught, knowledge of the level of conceptual and procedural development expected for a topic at a given school moment, and knowledge of the sequencing of various topics either within the same course or thinking about previous or subsequent courses, which are defined as follows:

1. *Knowledge that the teacher has about the mathematical contents that are required in the school grade* in which classes are being taught, this knowledge can be acquired by the teacher through a document that indicates what those contents are or what is the level of abstraction that students should develop in those grades. For example, in the Basic Learning Rights Volume 2, proposed by the Ministry of National Education (MEN, 2016) it is evidenced that students in grades one to three must have the ability to solve exercises of basic operations in mathematics, which should be known by the teacher who teaches mathematics in elementary basic education.
2. *Knowledge of the level of conceptual and procedural development expected for a topic at a given time in school* refers to knowledge about the depth with which a content should be approached according to a given school cycle. For example, knowing what type of classifications between polygons a student would be expected to make at the end of elementary school (fifth grade).
3. *Knowledge of the sequencing of various topics either within the same course or thinking about previous or subsequent courses* is the teacher's knowledge about the previous ideas or notions that a student has about a content in terms of what frames the curricular content, before knowing a topic to be developed. For example, the teacher's knowledge of the ideas of multiplication that his students have when they deepen this mathematical operation in the third grade.

Application of MTSK in Mathematics Education

The MTSK model has had a significant impact on the academic community, especially in Spain, where it emerged in 2013 and where it gained momentum. However, with the passage of time, several authors have deepened this model, adding new categories and characteristics to each of the subdomains of the domains that make up the model. In this order, since the year in which the model was established, there have been several articles, book chapters published by members of the SIDM group, doctoral theses directed by members of the aforementioned group and five Ibero-American Congresses on the specialized knowledge of the mathematics teacher; congress in which experiences are discussed around the use of the MTSK as a tool for the analysis of the teaching task, being the last congress of the MTSK, held in November 2021 in Brazil. In this way, these academic actions have contributed to form the Ibero-American MTSK Network, formed by members from the following countries: Spain, Portugal, Mexico, Costa Rica, Venezuela, Colombia, Ecuador, Peru, Chile, Brazil, Argentina and other European countries, such as Italy and Germany (Carrillo, 2019).

In this order of ideas, teachers in initial training, practicing teachers and teachers in continuing education (practicing teachers taking advanced studies) are the participants who have served as a sample for research on the characterization of their specialized knowledge. In the case of teachers in initial training, it has been explored about the specialized knowledge of teachers who are being trained to be Primary Basic Education teachers; however, in this model it is still a debt the exploration of the specialized knowledge of mathematics teachers who are being trained for Secondary or Middle Basic Education about the teaching of any mathematics content related to these levels of schooling, very despite Muñoz and Montes (2016) assure that in the context of secondary education this model is relevant due to the emphasis that is placed on the didactic-pedagogical knowledge of the contents, which in research processes has been scarcely explored, although it is very common to find in classrooms evidence of MTSK subdomains such as KMT and KFLM in teaching processes. As for practicing teachers, the opposite is the case of teachers in initial training. That is, being teachers who exercise their role and practice on a daily basis, they have been more explored with respect to their knowledge about the teaching of different mathematics contents.

Muñoz and Montes (2016) also highlight the importance of studying the MTSK model in the context of early childhood education teachers, since it allows them to give greater depth and mathematical rigor to the knowledge presented by teachers, since, despite the elementary nature of the concepts at this academic level, it is necessary for those who teach to have clarity about where, why and for what purpose what they teach arises. That is why one of the future challenges of this model is to expand to early childhood education teachers in mathematics, since in many cases teachers who teach at these levels of schooling do not have training in mathematics.

However, there are multiple different contents that have been studied according to the MTSK model. In the most recent Proceedings of the Seminar on Didactics of Science and Mathematics at the University of Huelva in 2017 and 2019, it was evidenced that the specialized knowledge of the teacher who teaches mathematics has been explored in topics such as: proportional magnitude problem solving (Barrera, Liñán and Pérez,2017); emotional and specialized knowledge of the mathematics teacher; specialized knowledge for teaching geometry in Early Childhood Education (Escudero-Domínguez, Muñoz-Catalán and Carrillo, 2017; Codes and Muñoz-Catalán, 2019; Escudero-Domínguez et al. , 2019); learning opportunities; mathematical knowledge mastery of MTSK also in early childhood education (Martín and Carrillo, 2017); affective mastery and MTSK (Pascual et al., 2019); the design of tasks for mathematics teacher training from the MTSK (Climent and Montes, 2019); complexification and simplification connections of the KMS in elementary education (Montañez-Esparza and Lizarde, 2019); and, the MTSK in the continuing education of mathematics teachers (Quiroga and Gamboa, 2017; Valenzuela-Molina and Ramos-Rodríguez, 2019). Thus, it is evident that this model has been expanding in each of the thoughts of mathematics, such as metric-geometric and variational thinking in countries such as Spain and Portugal in Europe, and in Brazil, Mexico and Chile in Latin America.

Conclusions

The MTSK model developed by Carrillo and his collaborators in 2013 and enhanced in 2018 has allowed the characterization of the specialized knowledge of the teacher who teaches mathematics at any academic level, both in mathematical and didactic-pedagogical knowledge, not explicitly evident in other models that address the knowledge of the mathematics teacher. This model supports through the categories presented for each subdomain of the mathematical and didactic-pedagogical domain, that there are characteristics of the mathematics graduate that differentiate them from other professionals who assume this role in Colombian schools, such as engineers, mathematicians, statisticians, physicists, and other types of professionals with training in mathematics. Likewise, it was evidenced that through this model it is possible to analyze the specialized knowledge of the topics, mathematical structures, mathematical practices, mathematics learning characteristics, mathematics teaching and mathematics learning standards of mathematics contents, and that given the breadth of the model allows the relationship between these subdomains of the same domain or of different domains.

In addition, it is concluded that the MTSK is an opportunity to study a line of research that relates the training of teachers with the specialized knowledge of teachers in initial training, since by analyzing their specialized knowledge in a certain subject of mathematics, there is the possibility of proposing improvement plans in the programs that train mathematics teachers. Therefore, this model is an invitation to teachers to reflect on their mathematical and didactic practices, since the same specificity offered by the MTSK is an opportunity to study new lines of research, such as the in-depth study of teachers' specialized knowledge in the teaching of mathematics at primary or higher levels using ICT, which have been scarcely addressed by means of the MTSK model as a theoretical reference.

Similarly, it is interesting that in Colombia the insertion of the specialized knowledge of the mathematics teacher continues to deepen as a topic of interest to be investigated in the training of the mathematics teacher. This is because we consider that it is a good opportunity to disseminate the need that in the

 ISSN (electronic): 2346-0806 ISSN (Print): 1794-8932

country for Mathematics graduates to be the ones in charge of teaching mathematics in schools, since this model describes in detail the knowledge that this type of professionals should have to teach mathematics contents..

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